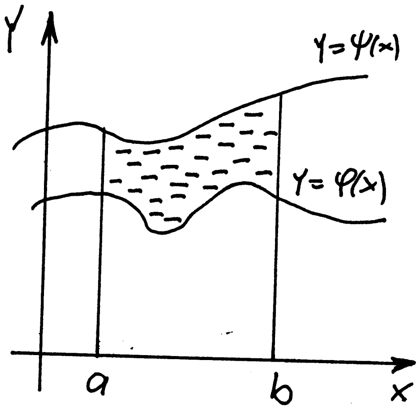
Trostruki integral

$$I = \iiint\limits_{\Omega} f(x,y,z)\,dx\,dy\,dz\,,$$ Ω oblast integracije u prostoru

ako je $\Omega: \begin{cases} a \le x \le b \\ \varphi(x) \le y \le \psi(x) \\ \alpha(x,y) \le z \le \beta(x,y) \end{cases}$ tada

$$I = \int_a^b dx \int_{\varphi(x)}^{\psi(x)} dy \int_{\alpha(x,y)}^{\beta(x,y)} f(x,y,z)\,dz$$

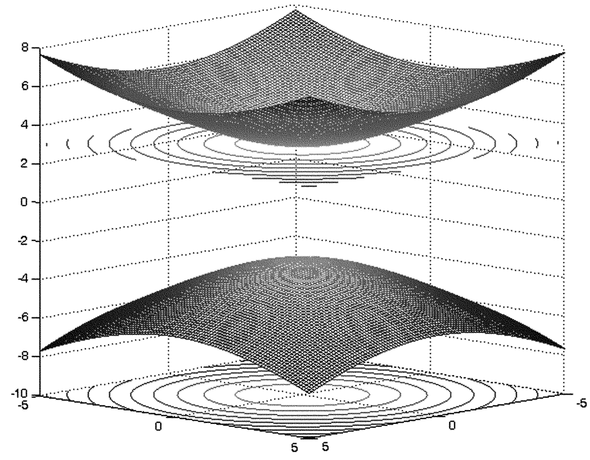


Oblast Ω možemo projicirati na

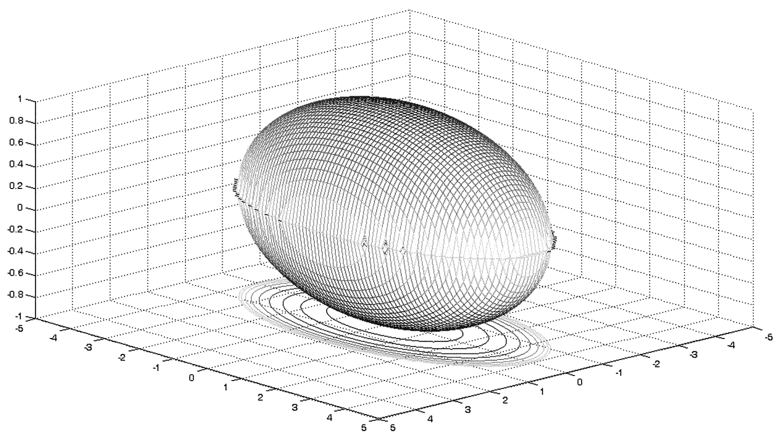
a) xoy ravan ili
b) yoz ravan ili
c) xoz ravan

U gornjem primjeru Ω smo prvo projicirali na xoy ravan.

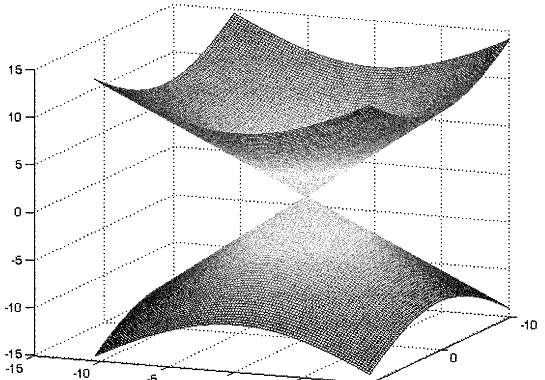
I se može izraziti na 6 načina.



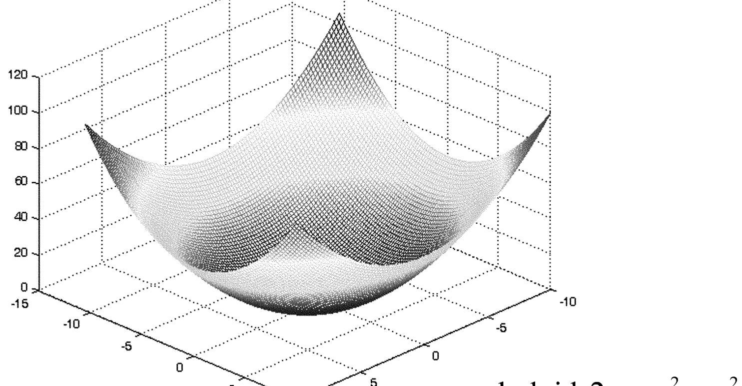
hiperboloid $x^2+y^2-z^2=-9$



elipsoid $\dfrac{x^2}{4}+\dfrac{y^2}{16}+\dfrac{z^2}{9}=1$



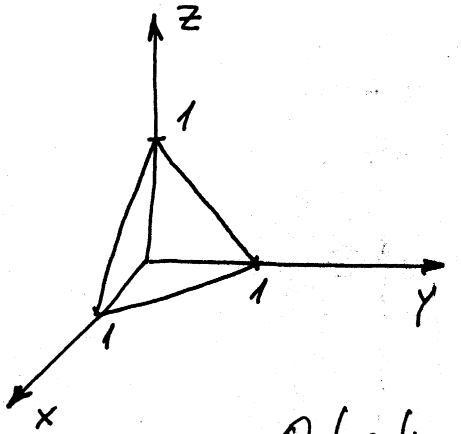
čunj $x^2+y^2=z^2$



paraboloid $2z=x^2+y^2$

(#) Izračunajte $\iiint\limits_{\Omega} (1-x)yz\,dx\,dy\,dz$ gdje je Ω oblast ograničena ravnima $x=0$, $y=0$, $z=0$ i $x+y+z=1$

Rj.



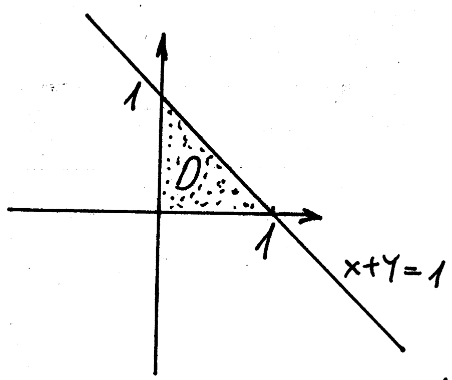
$x+y+z=1 \Leftrightarrow \frac{x}{1}+\frac{y}{1}+\frac{z}{1}=1$ segmentni oblik jedn. ravni

$x=0$ je yoz ravan

$y=0$ je xoz ravan

$z=0$ je xoy ravan

Odredimo projekciju oblasti na xoy ravan



$$\frac{\begin{array}{c} x+y+z=1 \\ z=0 \end{array}}{x+y=1}$$

$z=1-x-y$

Sa slike odredimo granice

$0 \le x \le 1$

$0 \le y \le 1-x$

$0 \le z \le 1-x-y$

$$\iiint\limits_{\Omega}(1-x)yz\,dx\,dy\,dz = \int_0^1 (1-x)\,dx \int_0^{1-x} y\,dy \int_0^{1-x-y} z\,dz = \int_0^1 (1-x)\,dx \int_0^{1-x} y\cdot\frac{1}{2}z^2\Big|_0^{1-x-y} dy$$

$$=\frac{1}{2}\int_0^1 (1-x)\,dx\int_0^{1-x} y\cdot(\underbrace{1-x}_{a}-\underbrace{y}_{b})^2\,dy = \frac{1}{2}\int_0^1 (1-x)\,dx\int_0^{1-x} y\left[(1-x)^2-2y(1-x)+y^2\right]dy$$

$$=\frac{1}{2}\int_0^1 (1-x)\,dx\int_0^{1-x}\left[(1-x)^2y-2y^2(1-x)+y^3\right]dy = \frac{1}{2}\int_0^1 (1-x)\left[(1-x)^2\frac{1}{2}y^2\Big|_0^{1-x} -\right.$$

$$\left.-2\cdot\frac{1}{3}y^3\Big|_0^{1-x}\cdot(1-x)+\frac{1}{4}y^4\Big|_0^{1-x}\right]dx = \frac{1}{2}\int_0^1 (1-x)\left[(1-x)^4\left(\frac{1}{2}-\frac{2}{3}+\frac{1}{4}\right)\right]dx$$

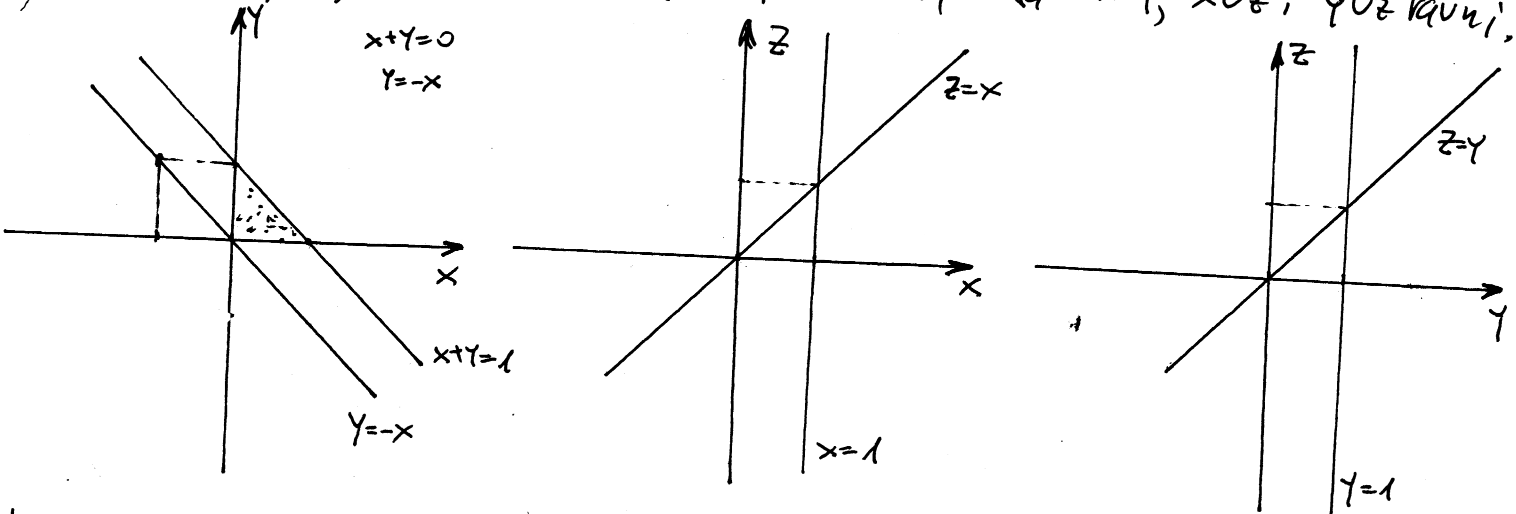
$$=\frac{1}{2}\cdot\frac{1}{12}\int_0^1 (1-x)^5\,dx = \left|\begin{array}{ll} 1-x=t & x=0 \Rightarrow t=1 \\ -dx=dt & x=1 \Rightarrow t=0 \\ dx=-dt & \end{array}\right| = -\frac{1}{24}\int_1^0 t^5\,dt = -\frac{1}{24}\cdot\frac{1}{6}t^6\Big|_1^0 = \frac{1}{144}$$

Izračunati trojni integral

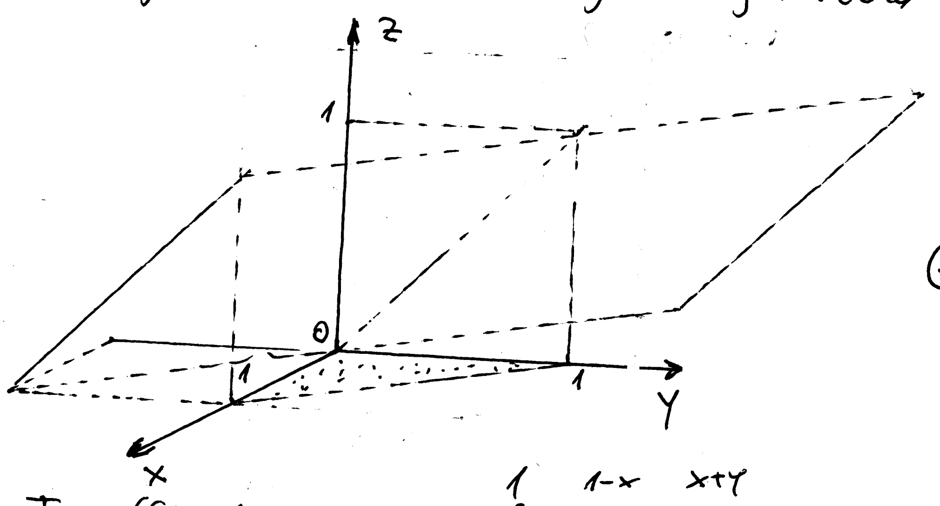
$$I=\iiint\limits_G \frac{1}{(1+z)^3}\,dx\,dy\,dz$$

gdje je oblast G u I oktantu ograničena ravnima $x+y=1$, $z=x+y$, $x=0$, $y=0$, $z=0$.

Rj. Napravimo presjek ravni $x+y=1$ i $z=x+y$ sa xOy, xOz i yOz ravni.



Iz presjeka vidimo da je ravan $x+y=1$ paralelna sa z osom a da je oblast G odozgo ograničena sa $z=x+y$ ravni



$$G:\begin{cases} 0\le x\le 1\\ 0\le y\le 1-x\\ 0\le z\le x+y\end{cases}$$

$$I=\iiint\limits_G \frac{1}{(1+z)^2}\,dx\,dy\,dz=\int_0^1 dx\int_0^{1-x} dy\int_0^{x+y}(1+z)^{-3}\,d(1+z)=\frac{1}{-2}\int_0^1 dx\int_0^{1-x}\overbrace{(1+z)^{-2}\Big|_0^{x+y}}^{(1+(x+y))^{-2}-1^{-2}}dy=$$

$$=-\frac{1}{2}\int_0^1 dx\int_0^{1-x}(1+x+y)^{-2}\,d(1+x+y)-\left(-\frac{1}{2}\right)\int_0^1 dx\int_0^{1-x} dy=-\frac{1}{2}\int_0^1(-1)(1+x+y)^{-1}\Big|_0^{1-x}dx+$$

$$+\frac{1}{2}\int_0^1 (1-x)\,dx = \frac{1}{2}\int_0^1 \left(2^{-1}-(1+x)^{-1}\right)dx + \frac{1}{2}\int_0^1 (1-x)\,dx =$$

$$\frac{1}{2} \quad -(1+x)^{-1}$$

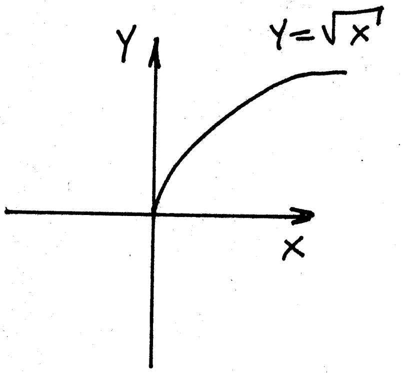
$$= \frac{1}{2}\left(\frac{1}{2}x\Big|_0^1 - \ln(1+x)\Big|_0^1 + x\Big|_0^1 - \frac{1}{2}x^2\Big|_0^1\right) =$$

$$= \frac{1}{2}\left(\frac{1}{2} - \ln 2 + 1 - \frac{1}{2}\right) = \frac{1}{2}(1-\ln 2)$$

traženo rješenje

#. Izračunati $I=\iiint\limits_{\Omega} y\cos(x+z)\,dx\,dy\,dz$ gdje je Ω oblast ograničena plohom $y=\sqrt{x}$ i ravnima $y=0$, $z=0$ i $x+z=\frac{\pi}{2}$.

Rj.

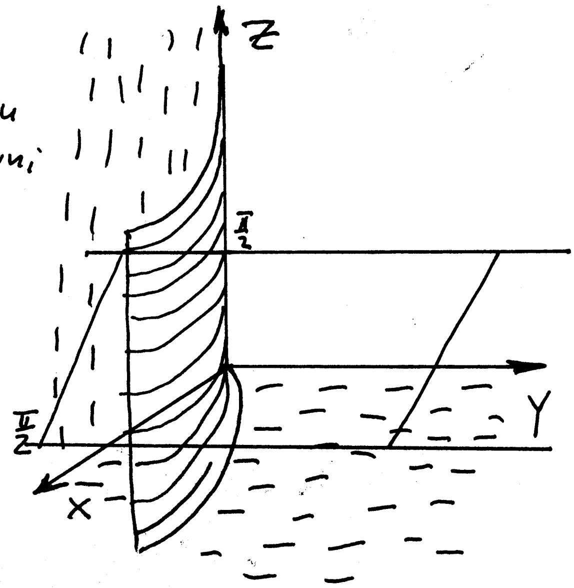


$y=0$ je xOz ravan

$z=0$ je xOy ravan

$x+z=\frac{\pi}{2}$

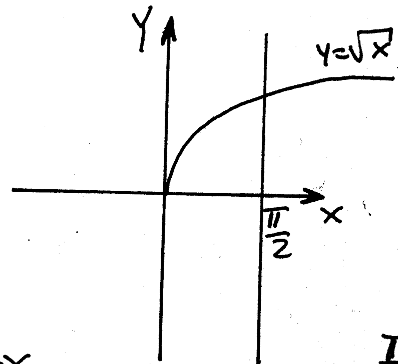
za $z=0$ dobiću projekciju ove ravni na xOy ravan



$0\le x\le \frac{\pi}{2}$

$0\le y\le \sqrt{x}$

$0\le z\le \frac{\pi}{2}-x$



$$I=\int_0^{\frac{\pi}{2}}dx\int_0^{\sqrt{x}}y\,dy\int_0^{\frac{\pi}{2}-x}\cos(x+z)\,dz=\int_0^{\frac{\pi}{2}}dx\int_0^{\sqrt{x}}y\sin(x+z)\Big|_{z=0}^{z=\frac{\pi}{2}-x}dy\overset{(*)}{=}$$

$$\boxed{\int\cos(x+a)\,dx=\left|\begin{matrix}x+a=t\\ dx=dt\end{matrix}\right|=\int\cos t\,dt=\sin t+c=\sin(x+a)+c}$$

$$\overset{(*)}{=}\int_0^{\frac{\pi}{2}}dx\int_0^{\sqrt{x}}y\Big[\underbrace{\sin\left(x+\frac{\pi}{2}-x\right)}_{1}-\sin x\Big]dy=\int_0^{\frac{\pi}{2}}\underbrace{\frac{1}{2}y^2\Big|_0^{\sqrt{x}}}_{x-0}(1-\sin x)\,dx=$$

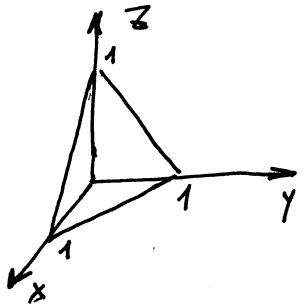
$$=\frac{1}{2}\int_0^{\frac{\pi}{2}}x\,dx-\frac{1}{2}\int_0^{\frac{\pi}{2}}x\sin x\,dx=\left|\begin{matrix}u=x & dv=\sin x\,dx\\ du=dx & v=-\cos x\end{matrix}\right|=\frac{1}{2}\cdot\frac{1}{2}x^2\Big|_0^{\frac{\pi}{2}}-$$

$$-\frac{1}{2}\left[-x\cos x\Big|_0^{\frac{\pi}{2}}+\int_0^{\frac{\pi}{2}}\cos x\,dx\right]=\frac{1}{4}\cdot\frac{\pi^2}{4}-\frac{1}{2}\sin x\Big|_0^{\frac{\pi}{2}}=\frac{\pi^2-8}{16}$$

(#) Izračunati trostruki integral $I=\iiint\limits_{\Omega}\frac{dx\,dy\,dz}{(x+y+z+1)^3}$,

ako je Ω oblast ograničena koordinatnim ravnima i ravni $x+y+z=1$.

Rj. $x+y+z=1$ je ravan koja na koordinatnim osama prolazi kroz tačke $(1,0,0)$, $(0,1,0)$ i $(0,0,1)$



$$\Omega:\begin{cases}0\le x\le 1\\ 0\le y\le 1-x\\ 0\le z\le 1-x-y\end{cases}$$

$$I=\int_0^1 dx\int_0^{1-x}dy\int_0^{1-x-y}\frac{dz}{(x+y+z+1)^3}\overset{(*)}{=}$$

$$\int\frac{dz}{(x+y+z+1)^2}=\left|\begin{matrix}x+y+z+1=t\\ dz=dt\end{matrix}\right|=\int\frac{dt}{t^3}=\int t^{-3}dt=\frac{t^{-2}}{-2}+C=$$

$$=\frac{-1}{2(x+y+z+1)^2}+C$$

$$\overset{(*)}{=}\int_0^1 dx\int_0^{1-x}\frac{-1}{2(x+y+z+1)^2}\Bigg|_0^{1-x-y}dy=\int_0^1 dx\int_0^{1-x}\Bigg(\frac{-1}{2(x+y+1-x-y+1)^2}-$$

$$-\frac{-1}{2(x+y+0+1)^2}\Bigg)dy=\int_0^1 dx\int_0^{1-x}\left(-\frac{1}{8}+\frac{1}{2(x+y+1)^2}\right)dy=$$

$$=-\frac{1}{2}\int_0^1 dx\int_0^{1-x}\left(\frac{1}{4}-\frac{1}{(x+y+1)^2}\right)dy\overset{(**)}{=}-\frac{1}{2}\int_0^1\left(\frac{1}{4}y\Big|_0^{1-x}+\frac{1}{x+y+1}\Big|_0^{1-x}\right)dx$$

$$\boxed{\int\frac{dy}{(x+y+1)^2}=\left|\begin{matrix}x+y+1=t\\ dy=dt\end{matrix}\right|=\int\frac{dt}{t^2}=\frac{t^{-1}}{-1}+C=\frac{-1}{t}+C=\frac{-1}{x+y+1}+C}\ \ldots(**)$$

$$=-\frac{1}{2}\int_0^1\left(\frac{1}{4}(1-x)+\frac{1}{2}-\frac{1}{x+1}\right)dx=-\frac{1}{2}\Bigg(\overbrace{\frac{1}{4}x\Big|_0^1-\frac{1}{4}\frac{x^2}{2}\Big|_0^1}^{\frac{1}{8}}+\frac{1}{2}x\Big|_0^1-\underbrace{\ln|x+1|\Big|_0^1}_{\ln 2-\ln 1}\Bigg)=\frac{1}{2}\ln 2-\frac{5}{16}$$

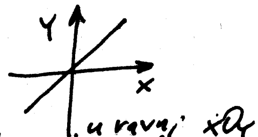
(#) Izračunati trostruki integral $I=\iiint\limits_{\Omega} z\,dx\,dy\,dz$, ako je

$\Omega: y=x,\ y=2x,\ 2x=1,\ x^2+y^2+z^2=1,\ z\geq 0$

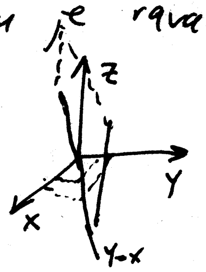
(oblast Ω je ograničena ovim površinama).

Rj: Komentarišimo površi koje čine Ω.

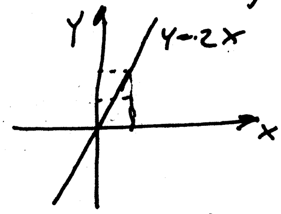
$y=x$ u ravni je prava



$y=x$ u prostoru je ravan koja sadrži pravu $y=x$

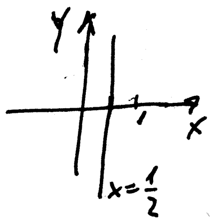


$y=2x$ u ravni je prava

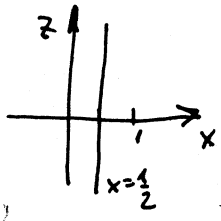


$y=2x$ u prostoru je ravan koja u ravni xOy sadrži pravu $y=2x$

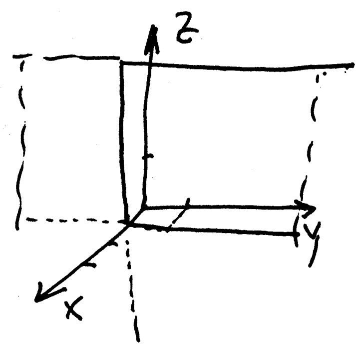
$2x=1$ u ravni xOy je prava



u ravni xOz je isto prava



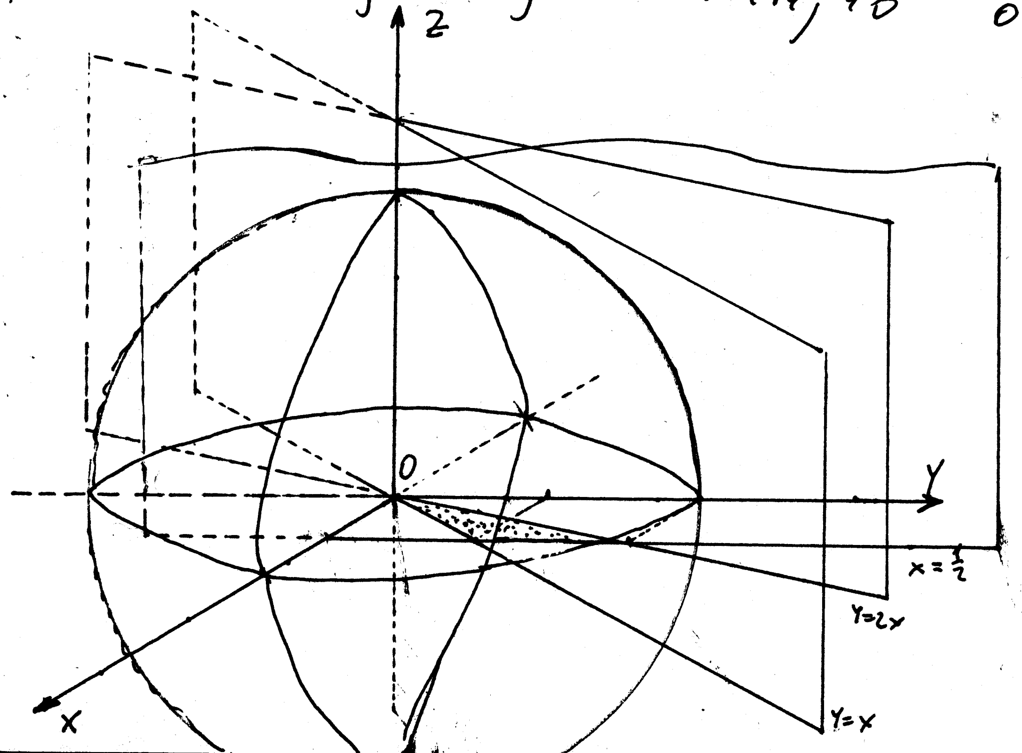
U prostoru to je ravan koja sadrži u xOz ravni pravu $x=\frac{1}{2}$ i u xOy ravni pravu $x=\frac{1}{2}$



$x=\frac{1}{2}$ je ravan koja je paralelna sa yOz osom

$x^2+y^2+z^2=1$ je jednačina kružnice

Na osnovu svega ovoga skicirajmo oblast Ω.



Oblast Ω je kružni isječak čija projekcija na xOy ravan je predstavljena tačkama na slici. Možemo zaključiti

$$\Omega: \begin{cases} 0 \leq x \leq \frac{1}{2} \\ x \leq y \leq 2x \\ 0 \leq z \leq \sqrt{1-x^2-y^2} \end{cases}$$

$$I=\iiint\limits_{\Omega} z\,dx\,dy\,dz = \int_0^{\frac12} dx \int_x^{2x} dy \int_0^{\sqrt{1-x^2-y^2}} z\,dz = \int_0^{\frac12} dx \int_x^{2x} \frac{1}{2} z^2\Big|_0^{\sqrt{1-x^2-y^2}} dy =$$

$$= \frac{1}{2} \int_0^{\frac12} dx \int_x^{2x} (1-x^2-y^2)\,dy = \frac{1}{2} \int_0^{\frac12} \left(y\Big|_x^{2x} - x^2 y\Big|_x^{2x} - \frac{1}{3} y^3\Big|_x^{2x} \right) dx =$$

$$= \frac{1}{2} \int_0^{1/2} \left(x - x^3 - \frac{1}{3}\, 7x^3\right) dx = \frac{1}{2} \int_0^{1/2} \left(x - \frac{10}{3} x^3\right) dx = \frac{1}{2}\left(\frac{1}{2}x^2\Big|_0^{\frac12} - \frac{\overset{5}{\cancel{10}}}{3}\cdot\frac{1}{\underset{2}{\cancel{4}}}x^4\Big|_0^{\frac12}\right)$$

$$= \frac{1}{2}\left(\frac{1}{2}\cdot\frac{1}{4} - \frac{5}{6}\cdot\frac{1}{16}\right) = \frac{1}{2}\left(\frac{1}{8} - \frac{5}{\ }\right) = \frac{1}{2}\cdot\frac{12-5}{96} = \frac{7}{192}$$

1. Izračunaj trostruki integral $I = \int_{-1}^{1} dx \int_{x^2}^{1} dy \int_{0}^{2} (4+z)dz$.

Rješenje:

$$I = \int_{-1}^{1} dx \int_{x^2}^{1} dy \int_{0}^{2} (4+z)dz = \int_{-1}^{1} dx \int_{x^2}^{1} 4z\Big|_0^2 + \frac{z^2}{2}\Big|_0^2 dy = \int_{-1}^{1} dx \int_{x^2}^{1} (8+2)dy = 10\int_{-1}^{1} y\Big|_{x^2}^{1} dx =$$

$$= 10\int_{-1}^{1} (1-x^2)dx = 10(x\Big|_{-1}^{1} - \frac{x^3}{3}\Big|_{-1}^{1}) = 10(2-\frac{2}{3}) = 10\cdot\frac{4}{3} = \frac{40}{3}$$

---

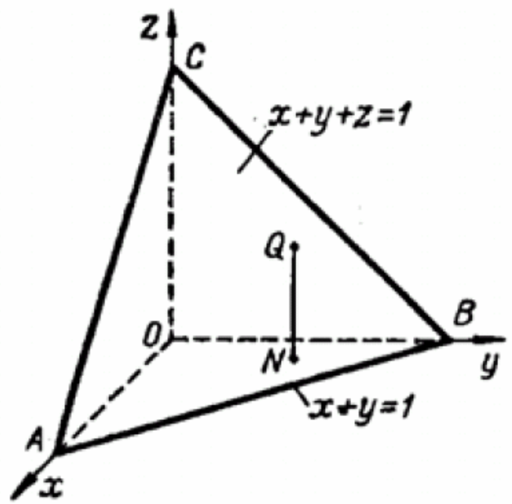
2. Izračunaj trostruki integral $\iiint\limits_G \frac{dxdydz}{1-x-y}$, gdje je G ograničena ravnima :

a) x+y+z=1, x=0, y=0, z=0;

b) x=0, x=1, y=2, y=5, z=2, z=4.

Rješenja:

a) $\iiint\limits_G \frac{dxdydz}{1-x-y}$ x=0, y=0, z=0



Skicirajmo oblast G (vidi sliku desno).

x+y+z=1 $\Leftrightarrow \frac{x}{1}+\frac{y}{1}+\frac{z}{1}=1$

x= 0 je yOz ravan
y= 0 je xOz ravan
z= 0 je xOy ravan

Odredimo projekciju oblasti na xOy ravan:
Nacrtati sliku (uputa: pogledati xoy ravan sa slike desno).

x+y+z=1
z= 0

x+y=1
z= 1-x-y

Sa slike projekcije odredimo granice: $\begin{array}{l} 0 \le x \le 1 \\ 0 \le y \le 1-x \\ 0 \le z \le 1-x-y \end{array}$

$$\iiint\limits_G \frac{dxdydz}{1-x-y} = \int\limits_0^1 dx \int\limits_0^{1-x} \frac{dy}{1-x-y} \int\limits_0^{1-x-y} dz = \int\limits_0^1 dx \int\limits_0^{1-x} \left(\frac{1}{1-x-y} \cdot z \Big|_0^{1-x-y} \right) dy =$$

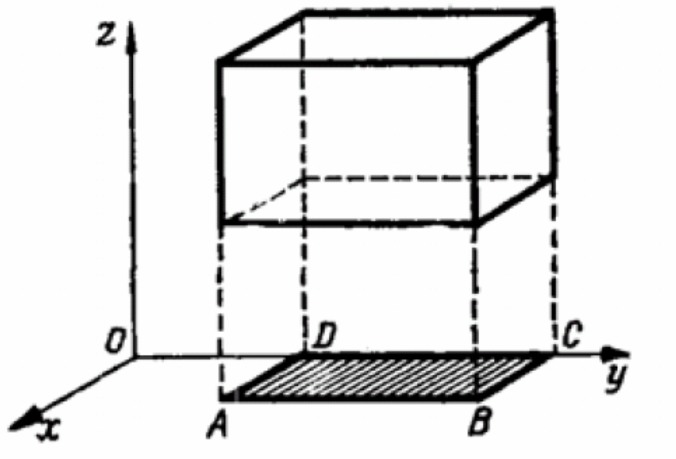
$$= \int\limits_0^1 dx \int\limits_0^{1-x} \left(\frac{1}{1-x-y} \cdot (1-x-y)\right) dy = \int\limits_0^1 dx \int\limits_0^{1-x} dy = \int\limits_0^1 y\Big|_0^{1-x} dx = \int\limits_0^1 (1-x) dx =$$

$$= x\Big|_0^1 - \frac{x^2}{2}\Big|_0^1 = 1 - \frac{1}{2} = \frac{1}{2}$$

---

b) $\iiint\limits_G \frac{dxdydz}{1-x-y}$ x=0, x=1, y=2, y=5, z=2, z=4.

Skicirajmo oblast G (vidi sliku).



$$\int\limits_0^1 dx \int\limits_2^5 \frac{dy}{1-x-y} \int\limits_2^4 dz = \int\limits_0^1 dx \int\limits_2^5 z\Big|_2^4 \frac{dy}{1-x-y} = 2\int\limits_0^1 dx \int\limits_2^5 \frac{dy}{1-x-y} = \left| \begin{array}{l} 1-x-y=t \\ -dy=dt \\ y=2 \Rightarrow t=-1-x \\ y=5 \Rightarrow t=-4-x \end{array} \right| =$$

$$-2\int\limits_0^1 dx \int\limits_{-1-x}^{-4-x} \frac{dt}{t} = -2\int\limits_0^1 \ln|t| \Big|_{-1-x}^{-4-x} = -2\int\limits_0^1 \{\ \ln[-(4-x)] - \ln(-1-x)\ \} dx =$$

$$= -2\int\limits_0^1 \ln|x+4| dx + 2\int\limits_0^1 \ln|x+1| dx \overset{*}{=}$$

Zadaci za vježbu

U zadacima **3474. — 3476.** **proceniti date integrale.**

3474. $\iiint\limits_{\Omega} (x^2+y^2+z^2)\,dv$, gde je Ω—lopta $x^2+y^2+z^2<R^2$.

3475. $\iiint\limits_{\Omega} (x+y+z)\,dv$, gde je Ω—lopta $x\geqslant 1$, $y>1$, $z>1$, $x<3$, $y<3$, $z<3$.

3476. $\iiint\limits_{\Omega} (x+y-z+10)\,dv$, gde je Ω—lopta $x^2+y^2+z^2<3$.

U zadacima 3517 — 3524 izračunati navedene trostruke i trojne integrale

3517. $\int\limits_0^1 dx \int\limits_0^2 dy \int\limits_0^3 dz.$ **3518.** $\int\limits_0^a dx \int\limits_0^b dy \int\limits_0^c (x+y+z)\,dz.$

3519. $\int\limits_0^a dx \int\limits_0^x dx \int\limits_0^y xyz\,dz.$ **3520.** $\int\limits_0^a dx \int\limits_0^x dy \int\limits_0^{xy} x^3y^2z\,dz.$

3521. $\int\limits_0^{e-1} dx \int\limits_0^{e-x-1} dy \int\limits_e^{x+y+e} \frac{\ln(z-x-y)}{(x-e)(x+y-e)}\,dz.$

3522. $\iiint\limits_{\Omega} \frac{dx\,dy\,dz}{(x+y+z+1)^3}$, Ω je oblast ograničena ravnima $x=0$, $y=0$, $z=0$, $x+y+z=1$.

3523. $\iiint\limits_{\Omega} xy\,dx\,dy\,dz$, Ω je oblast ograničena hiperboličnim paraboloidom $z=xy$ i ravnima $x+y=1$ i $z=0$ $(z\geqslant 0)$.

3524. $\iiint\limits_{\Omega} y\cos(z+x)\,dx\,dy\,ds$, Ω je oblast ograničena cilindrom $y=\sqrt{x}$ i ravnima $y=0$, $z=0$ i $x+z=\frac{\pi}{2}$.

Rješenja

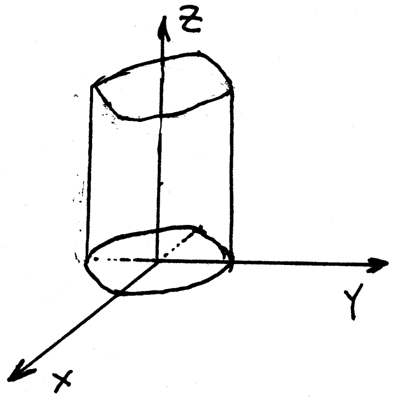
3474. $0<I<\frac{4}{3}\pi R^5$. **3475.** $24<I<72$.

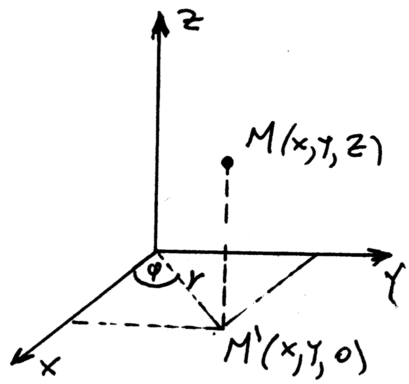
3476. $29\pi\sqrt{3}<I<52\pi\sqrt{3}$. **3517.** 6. **3518.** $\frac{abc(a+b+c)}{2}$. **3519.** $\frac{a^6}{48}$. **3520.** $\frac{a^{11}}{110}$.

3521. $2e-5$. **3522.** $\frac{1}{2}\left(\ln 2-\frac{5}{8}\right)$. **3523.** $\frac{1}{180}$. **3524.** $\frac{\pi^2}{16}-\frac{1}{2}$.

Računanje trostrukih integrala uvođenjem cilindričnih i sfernih koordinata

cilindrične koordinate





uvodimo smjenu

$x = r\cos\varphi$

$y = r\sin\varphi$

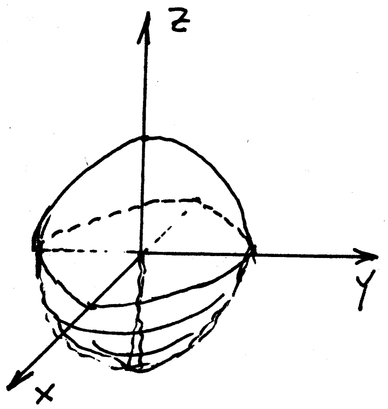
$z = z$

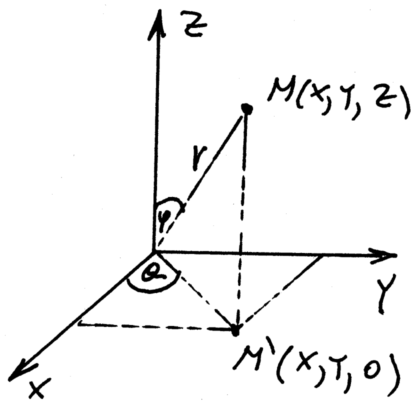
$dx\,dy\,dz = r\,dr\,d\varphi\,dz$

cilindrične koordinate obično uvedemo ako se pojavi izraz x^2+y^2 $\left(x^2+y^2 = r^2\cos^2\varphi + r^2\sin^2\varphi = r^2(\sin^2\varphi+\cos^2\varphi) = r^2\right)$

$(r \geq 0,\quad 0 \leq \varphi \leq 2\pi)$

sferne koordinate





uvodimo smjenu

$x = r\sin\varphi\cos\theta$

$y = r\sin\varphi\sin\theta$

$z = r\cos\varphi$

$dx\,dy\,dz = r^2\sin\varphi\,dr\,d\varphi\,d\theta$

$r \geq 0$

$0 \leq \varphi \leq \pi$

$0 \leq \theta \leq 2\pi$

$$x^2+y^2+z^2 = r^2\sin^2\varphi\cos^2\theta + r^2\sin^2\varphi\sin^2\theta + r^2\cos^2\varphi = \ldots = r^2$$

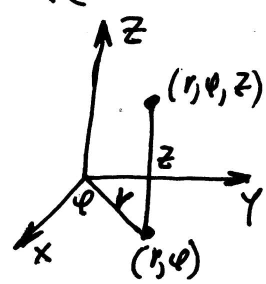
sferne koordinate obično uvodimo ako se u podintegralnoj f-ji ili u opisu oblasti integracije pojavljuje izraz $x^2+y^2+z^2$.

(#) Dati trojni integral $\iiint\limits_{\Omega} f(x,y,z)\,dx\,dy\,dz$ transformisati na trostruki u cilindričnim koordinatama (sa određenim posebnim granicama integracije) ako je Ω oblast u prvom oktantu ograničen cilindrom $x^2+y^2=R^2$ i ravnima $z=0$, $z=1$, $y=x$ i $y=x\sqrt{3}$

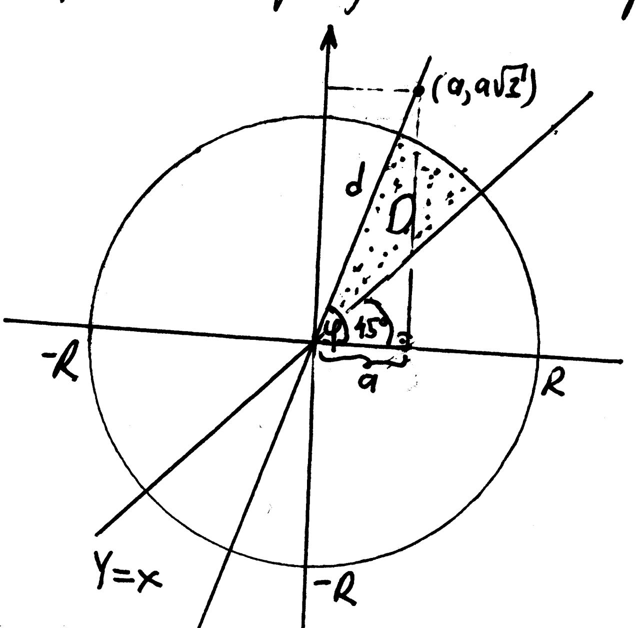
Rj.

Cilindrične koordinate glase

$$x = r\cos\varphi$$
$$y = r\sin\varphi$$
$$z = z$$
$$dx\,dy\,dz = r\,dr\,d\varphi\,dz$$



Napravimo presjek datih površina sa xOy ravni.



$$\cos\varphi = \frac{a}{d} = \frac{a}{2a} = \frac{1}{2} \Rightarrow \varphi = \frac{\pi}{3}$$
$$d^2 = a^2 + 3a^2 = 4a^2$$
$$d = 2a$$

Sad nije teško vidjeti da je

$$\iiint\limits_{\Omega} f(x,y,z)\,dx\,dy\,dz = \int_0^1 dz \int_{\pi/4}^{\pi/3} d\varphi \int_0^R f(r\cos\varphi, r\sin\varphi, z)\, r\,dr.$$

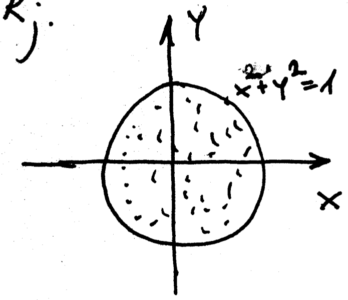
$$\Omega \xrightarrow{\text{transformiše}} \Omega'$$

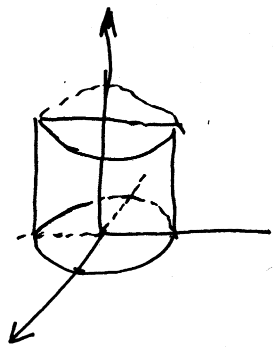
$$\Omega' : \begin{cases} 0 \le r \le R \\ \frac{\pi}{4} \le \varphi \le \frac{\pi}{3} \\ 0 \le z \le 1 \end{cases}$$

(#) Izračunati $I=\iiint\limits_{\Omega}(x^2+y^2+z)^3\,dx\,dy\,dz$ gdje je

Ω oblast ograničena sa $x^2+y^2=1$, $z=0$ i $z=1$.

Rj.





uvodimo smjene:

$x=r\cos\varphi$

$y=r\sin\varphi$

$z=z$

$x^2+y^2=r^2\cos^2\varphi+r^2\sin^2\varphi=r^2$

$x^2+y^2=1$ $\quad 0\le z\le 1$

$r^2=1$ $\quad 0\le\varphi\le 2\pi$

$r\ge 0$

$0\le r\le 1$

$$\Omega'=\begin{cases}0\le r\le 1\\ 0\le\varphi\le 2\pi\\ 0\le z\le 1\end{cases}$$

$dx\,dy\,dz=$
$=r\,dr\,d\varphi\,dz$

$$I=\iiint\limits_{\Omega}(x^2+y^2+z)^3\,dx\,dy\,dz=\iiint\limits_{\Omega'}(r^2+z)^3 r\,dr\,d\varphi\,dz=$$

$$=\int_0^1 dz\int_0^{2\pi}d\varphi\int_0^1 (r^2+z)^3\cdot r\,dr=\left|\begin{matrix}r^2+z=t & r=0\Rightarrow t=z\\ 2r\,dr=dt & r=1\Rightarrow t=z+1\\ r\,dr=\frac{1}{2}dt & \end{matrix}\right|$$

$$=\frac{1}{2}\int_0^1 dz\int_0^{2\pi}d\varphi\int_z^{z+1}t^3\,dt=\frac{1}{2}\int_0^1 dz\int_0^{2\pi}\frac{1}{4}t^4\Big|_z^{z+1}d\varphi=\frac{1}{8}\int_0^1\left[(z+1)^4-z^4\right]\varphi\Big|_0^{2\pi}dz$$

$$=\frac{1}{8}\cdot 2\pi\int_0^1\left[(z+1)^4-z^4\right]dz=\frac{\pi}{4}\cdot\left(\frac{1}{5}(z+1)^5\Big|_0^1-\frac{1}{5}z^5\Big|_0^1\right)=$$

$$\boxed{\int(z+1)^4\,dz=\left|\begin{matrix}t=z+1\\ dt=dz\end{matrix}\right|=\int t^4\,dt=\frac{1}{5}t^5+c=\frac{1}{5}(z+1)^5+c}$$

$$=\frac{\pi}{20}(31-1)=\frac{30\pi}{20}=\frac{3\pi}{2}$$

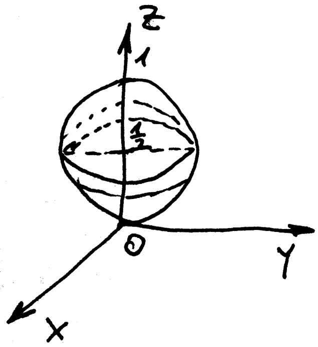
Izračunati $I=\iiint\limits_{\Omega}\sqrt{x^2+y^2+z^2}\,dx\,dy\,dz$ gdje je Ω oblast ograničena sferom $x^2+y^2+z^2=z$.

Rj.

$$x^2+y^2+z^2=z$$
$$x^2+y^2+z^2-z=0$$
$$x^2+y^2+z^2-2\cdot z\cdot\frac{1}{2}+\left(\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2=0$$
$$x^2+y^2+\left(z-\frac{1}{2}\right)^2=\frac{1}{4}$$

centar sfere u tački $S(0,0,\frac{1}{4})$

poluprečnik sfere $r=\frac{1}{2}$



uvodimo smjene

$$x=r\sin\varphi\cos\theta$$
$$y=r\sin\varphi\sin\theta$$
$$z=r\cos\varphi$$

Odredimo granice za r, φ i θ nove oblasti

$$x^2+y^2+z^2=r^2$$

$r^2=r\cos\varphi$ $\nearrow$ iz $x^2+y^2+z^2=z$

$/:r \quad (r\neq 0)$

$r=\cos\varphi$ kako je $r>0 \Rightarrow \cos\varphi>0$ tj. $0\leq\varphi\leq\frac{\pi}{2}$

$$\Omega':\begin{cases}0\leq r\leq\cos\varphi\\ 0\leq\varphi\leq\frac{\pi}{2}\\ 0\leq\theta\leq 2\pi\end{cases}\qquad dx\,dy\,dz=r^2\sin\varphi\,dr\,d\varphi\,d\theta$$

$$I=\iiint\limits_{\Omega}\sqrt{x^2+y^2+z^2}\,dx\,dy\,dz=\iiint\limits_{\Omega'}\sqrt{r^2}\,r^2\sin\varphi\,dr\,d\varphi\,d\theta=$$

$$\int_0^{2\pi}d\theta\int_0^{\frac{\pi}{2}}\sin\varphi\,d\varphi\int_0^{\cos\varphi}r^3\,dr=\int_0^{2\pi}d\theta\int_0^{\pi/2}\sin\varphi\,\frac{1}{4}r^4\Big|_0^{\cos\varphi}d\varphi=$$

$$=\frac{1}{4}\int_0^{2\pi}d\theta\int_0^{\pi/2}\sin\varphi\cos^4\varphi\,d\varphi=\left|\begin{matrix}\cos\varphi=t & \varphi=0\Rightarrow t=1\\ -\sin\varphi\,d\varphi=dt & \varphi=\frac{\pi}{2}\Rightarrow t=0\\ \sin\varphi\,d\varphi=-dt & \end{matrix}\right|=$$

$$=\frac{1}{4}\int_0^{2\pi}d\theta\int_0^1 t^4\,dt=\frac{1}{4}\int_0^{2\pi}\frac{1}{5}t^5\Big|_0^1\,d\theta=\frac{1}{20}\,\theta\Big|_0^{2\pi}=\frac{1}{20}\cdot 2\pi=\frac{\pi}{10}$$

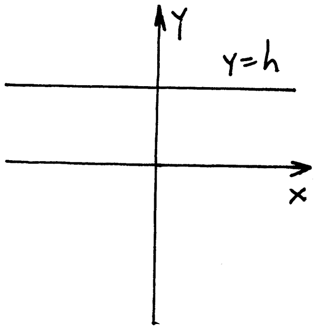
Izračunati trostruki integral

$$K=\iiint\limits_T y\,dx\,dy\,dz$$

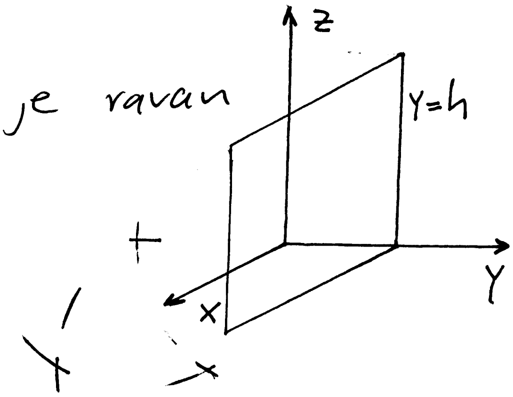
gdje je oblast T ograničena površinama $y=\sqrt{x^2+z^2}$ i $y=h$, $h>0$.

Rj. Pokušajmo skicirati oblast T.

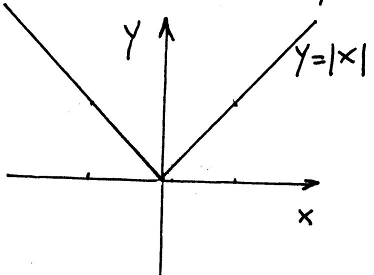
U xOy-ravni $y=h$ je prava.



U prostoru $y=h$ je ravan

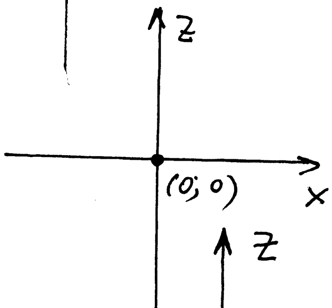


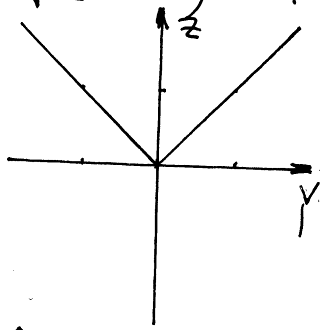
U xOy-ravni površina $y=\sqrt{x^2+z^2}$ je oblika $y=\sqrt{x^2}$



U xOz-ravni površina $y=\sqrt{x^2+z^2}$ je oblika $0=\sqrt{x^2+z^2}$ tj. tačka $(0;0)$.

U yOz-ravni površina $y=\sqrt{x^2+z^2}$ je oblika $y=\sqrt{z^2}$ tj. $y=|z|$.

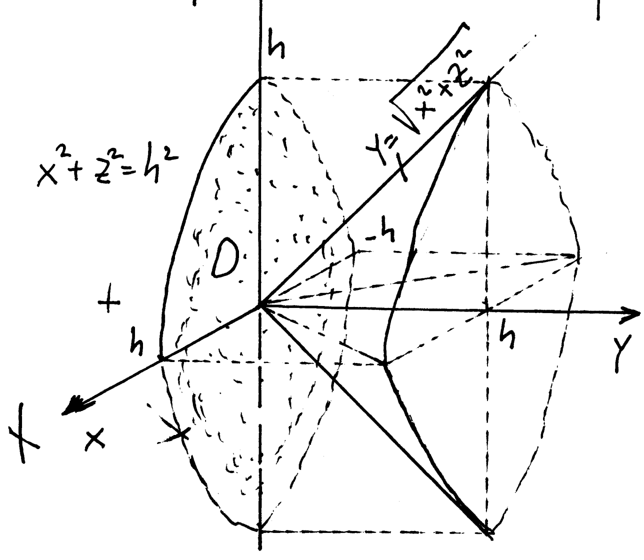




Ako napravimo presjek površina $y=\sqrt{x^2+z^2}$ i $y=h$ dobićemo $h=\sqrt{x^2+z^2}$ tj.

$$x^2+z^2=h^2$$

(krug poluprečnika h)



Oblast T (pola čunja) je prikazan na slici lijevo. Ako napravimo projekciju oblasti T na xOz ravan dobićemo sljedeće granice:

$$T:\begin{cases} -h \le x \le h \\ -\sqrt{h^2-z^2} \le z \le \sqrt{h^2-z^2} \\ \sqrt{x^2+z^2} \le y \le h \end{cases}$$

Pomoću pravougaonih koordinata dati trostruki integral je teško izračunati.

Uvodimo cilindrične koordinate i to

$$x = r\cos\varphi$$
$$z = r\sin\varphi$$
$$y = y$$
$$dx\,dy\,dz = r\,dr\,d\varphi\,dy$$

$$T \xrightarrow{\text{transformiše}} T' : \begin{cases} 0 \le r \le h \\ 0 \le \varphi \le 2\pi \\ r \le y \le h \end{cases}$$

Prema tome

$$K = \iiint_T y\,dx\,dy\,dz = \left| \begin{matrix} \text{uvodimo} \\ \text{cilindrične} \\ \text{koordinate} \end{matrix} \right| = \iiint_{T'} y\,r\,dr\,d\varphi\,dy =$$

$$= \int_0^{2\pi} d\varphi \int_0^h r\,dr \int_r^h y\,dy = \int_0^{2\pi} d\varphi \int_0^h r\,\underbrace{\frac{1}{2}y^2\Big|_r^h}_{h^2-r^2}\,dr =$$

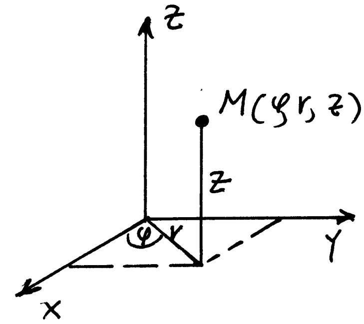
$$= \frac{1}{2}\int_0^{2\pi} d\varphi \int_0^h (rh^2 - r^3)\,dr = \frac{1}{2}\int_0^{2\pi} \Big(\underbrace{\frac{1}{2}r^2h^2\Big|_0^h}_{\frac{1}{2}r^4} \underbrace{- \frac{1}{4}r^4\Big|_0^h}_{-\frac{1}{4}h^4} \Big)\,d\varphi$$

$$= \frac{1}{2}\cdot\frac{1}{4}h^4 \int_0^{2\pi} d\varphi = \frac{1}{8}h^4\,\varphi\Big|_0^{2\pi} = \frac{h^4\pi}{4}$$

traženo rješenje

(#) Dat je trostruki integral $\int_0^{2\pi} d\varphi \int_0^2 r^3 dr \int_0^{\sqrt{4-r^2}} dz$ u cilindričnim koordinatama. Skicirati oblast integracije i izračunati taj integral prelazeći na sferne koordinate.

Rj: U cilindričnim koordinatama proizvoljna tačka M je opisana na sljedeći način



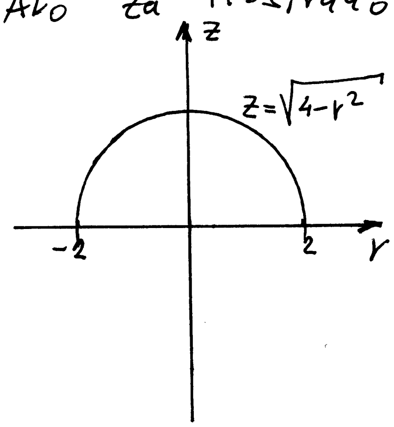
$$\Omega: \begin{cases} 0 \le \varphi \le 2\pi \\ 0 \le r \le 2 \\ 0 \le z \sqrt{4-r^2} \end{cases}$$

Na osnovu izgleda oblasti Ω vidimo da je projekcija figure na xOy ravan oblika

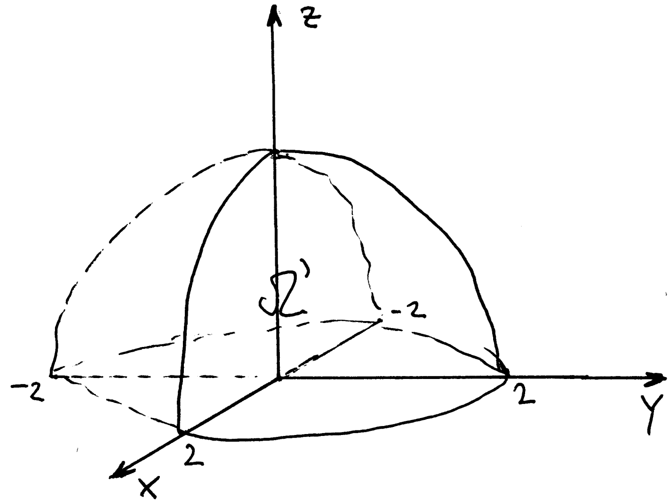
$$D: \begin{cases} 0 \le \varphi \le 2\pi \\ 0 \le r \le 2 \end{cases}$$

tj. krug sa centrom u koordinatnom početku poluprečnika 2.

Ako za fiksirano φ posmatramo rOz ravan imamo



Prema tome oblast integracije Ω je polulopta



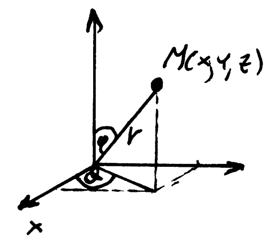
Cilindrične koordinate glase

$x = r\cos\varphi$
$y = r\sin\varphi$
$z = z$
$dx\,dy = r\,dr\,d\varphi$

Tako da bi prelaskom na pravougaone koordinate sad imali

$$\int_0^{2\pi} d\varphi \int_0^{2} r^3\,dr \int_0^{\sqrt{1-r^2}} dz = \iiint_{\Omega} r^2\, r\,dr\,d\varphi\,dz = \left|\begin{array}{l} \text{prelazimo na pravougaone} \\ \text{koordinate} \\ \Omega \xrightarrow{\text{transformiše}} \Omega' \\ r\,dr\,d\varphi = dx\,dy \\ r^2 = r^2(\sin^2\varphi + \cos^2\varphi) \\ \quad = r^2\sin^2\varphi + r^2\cos^2\varphi \\ \quad = (r\sin\varphi)^2 + (r\cos\varphi)^2 \\ \quad = x^2+y^2 \end{array}\right| =$$

$$= \iiint_{\Omega'} (x^2+y^2)\,dx\,dy\,dz$$



Sferne koordinate glase

$$x = r\sin\varphi\cos\theta$$
$$y = r\sin\varphi\sin\theta$$
$$z = r\cos\varphi$$
$$dx\,dy\,dz = r^2\sin\varphi\,dr\,d\varphi\,d\theta$$

$$\Omega' \xrightarrow{\text{transformiše}} \Omega'' : \begin{cases} 0 \le r \le 2 \\ 0 \le \varphi \le \pi/2 \\ 0 \le \theta \le 2\pi \end{cases} \qquad x^2+y^2 = r^2\sin^2\varphi$$

$$\iiint_{\Omega'} (x^2+y^2)\,dx\,dy\,dz = \left|\begin{array}{l} \text{uvodimo} \\ \text{sferne} \\ \text{koordinate} \end{array}\right| = \iiint_{\Omega''} r^2\sin^2\varphi\; r^2\sin\varphi\,dr\,d\theta\,d\varphi =$$

$$= \int_0^{2\pi} d\theta \int_0^{2} r^4\,dr \int_0^{\pi/2} \sin^3\varphi\,d\varphi \overset{(*)}{=} \theta\Big|_0^{2\pi} \cdot \frac{1}{5}r^5\Big|_0^{2} \cdot \frac{2}{3} = \frac{2^7}{15}\pi$$

traženo rješenje

$$\int_0^{\pi/2} \sin^3\varphi\,d\varphi = \int_0^{\pi/2} \sin\varphi\,\underbrace{(1-\cos^2\varphi)}_{\sin^2\varphi}\,d\varphi = \left|\begin{array}{l} d(\sin\varphi) = \cos\varphi\,d\varphi \\ d(\cos\varphi) = -\sin\varphi\,d\varphi \end{array}\right| = -\int_0^{\pi/2} (1-\cos^2\varphi)\,d(\cos\varphi)$$

$$= -\left(\cos\varphi\Big|_0^{\pi/2} - \frac{1}{3}\cos^3\varphi\Big|_0^{\pi/2}\right) = -\left((0-1) - \frac{1}{3}(0-1)\right) = -\left(-1+\frac{1}{3}\right) = \frac{2}{3} \quad \ldots(*)$$

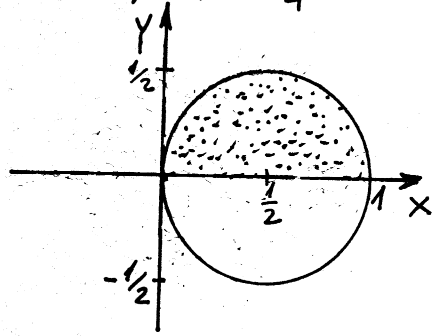
(#) Izračunati integral $\iiint\limits_{\Omega} \sqrt{z\cdot(x^2+y^2)}\,dx\,dy\,dz$ gdje je Ω oblast $x^2+y^2 \leq x$; $y \geq 0$, $z \geq 0$, $z \leq 3$.

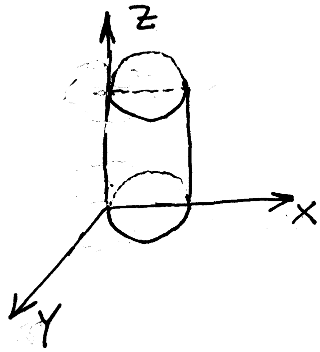
Rj. U ravni xOy kako izgleda $x^2+y^2 \leq x$?

$x^2 - x + y^2 = 0$

$x^2 - 2\cdot\frac{1}{2}\cdot x + \frac{1}{4} + y^2 = \frac{1}{4}$

$(x-\frac{1}{2})^2 + y^2 = \frac{1}{4}$





Uvodimo smjenu

$x = r\cos\varphi$

$y = r\sin\varphi$

$z = z$

$x^2+y^2 \leq x$

$r^2\cos^2\varphi + r^2\sin^2\varphi \leq r\cos\varphi$

$r^2 \leq r\cos\varphi \quad /:r \quad (r \neq 0)$

$r \leq \cos\varphi$

kako je $r \geq 0$ to je $\cos\varphi \geq 0$

$y \geq 0$

$r\sin\varphi \geq 0 \quad /:r$

$\sin\varphi \geq 0$

imam $0 \leq r \leq \cos\varphi$

$\sin\varphi \geq 0$

$\cos\varphi \geq 0$

$0 \leq z \leq 3$

$$\Rightarrow \quad \Omega' : \begin{cases} 0 \leq r \leq \cos\varphi \\ 0 \leq \varphi \leq \frac{\pi}{2} \\ 0 \leq z \leq 3 \end{cases} \qquad dx\,dy\,dz = J\,d\varphi\,dr\,dz$$

$$\iiint\limits_{\Omega} \sqrt{z(x^2+y^2)}\,dx\,dy\,dz = \iiint\limits_{\Omega'} \sqrt{zr^2}\,r\,d\varphi\,dr\,dz = \int_0^3 dz \int_0^{\frac{\pi}{2}} d\varphi \int_0^{\cos\varphi} \sqrt{z}\,r^2\,dr =$$

$$= \int_0^3 \sqrt{z}\,dz \int_0^{\frac{\pi}{2}} \frac{1}{3} r^3\Big|_0^{\cos\varphi} d\varphi = \frac{1}{3}\int_0^3 \sqrt{z}\,dz \int_0^{\frac{\pi}{2}} \cos^3\varphi\,d\varphi = \frac{1}{3}\int_0^3 \sqrt{z}\,dz \int_0^{\frac{\pi}{2}} \cos\varphi(1-\sin^2\varphi)\,d\varphi$$

$$= \left| \begin{matrix} \sin\varphi = t & \varphi = 0 \Rightarrow t = 0 \\ \cos\varphi\,d\varphi = dt & \varphi = \frac{\pi}{2} \Rightarrow t = 1 \end{matrix} \right| = \frac{1}{3}\int_0^3 \sqrt{z}\,dz \int_0^1 (1-t^2)\,dt = \frac{1}{3}\int_0^3 \sqrt{z}\left(t\Big|_0^1 - \frac{1}{3}t^3\Big|_0^1\right)dz$$

$$= \frac{1}{3}\left(1-\frac{1}{3}\right)\frac{z^{\frac{3}{2}}}{\frac{3}{2}}\Bigg|_0^3 = \frac{1}{3}\cdot\frac{2}{3}\cdot\frac{2}{3}\,3^{\frac{3}{2}} = \frac{4}{27}\sqrt{3^3} = \frac{4}{9}\sqrt{3}$$

(#) Izračunati trostruki integral $J=\iiint\limits_{W}(x^2+y^2+z^2)\,dx\,dy\,dz$

gdje je oblast W ograničena površinom $3(x^2+y^2)+z^2=3a^2$.

Rj. Skicirajmo oblast W

$$3(x^2+y^2)+z^2=3a^2$$

$$3x^2+3y^2+z^2=3a^2 \quad /:3a^2$$

$$\frac{x^2}{a^2}+\frac{y^2}{a^2}+\frac{z^2}{3a^2}=1$$

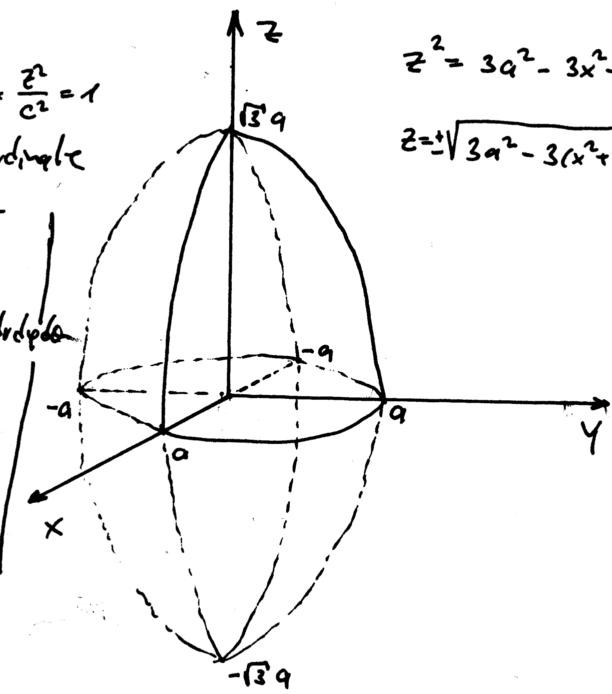
jednačina elipse

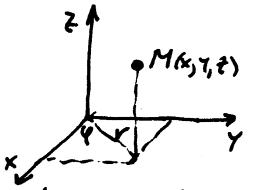
za elipsu $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ uopštene sferne koordinate glase
$x=ar\sin\psi\cos\theta$
$y=br\sin\psi\sin\theta$
$z=cr\cos\psi$
$dx\,dy\,dz=abc\,r^2\sin\psi\,dr\,d\psi\,d\theta$

U ovom slučaju uopštene sferne koordinate ne mogu na lagan način riješit zadatak

$z^2=3a^2-3x^2-3y^2$

$z=\pm\sqrt{3a^2-3(x^2+y^2)}$





Uvodimo cilindrične koordinate

$$x=r\cos\varphi$$
$$y=r\sin\varphi$$
$$z=z$$
$$dx\,dy\,dz=r\,dr\,d\varphi\,dz$$

$$x^2+y^2+z^2=r^2+z^2$$

$$W \xrightarrow{\text{transformiše}} W':\begin{cases}0\le r\le a\\ 0\le\varphi\le 2\pi\\ -\sqrt{3(a^2-(x^2+y^2))}\le z\le\sqrt{3(a^2-(x^2+y^2))}\\ -\sqrt{3(a^2-r^2)}\le z\le\sqrt{3(a^2-r^2)}\end{cases}$$

$$J=\iiint\limits_{W}(x^2+y^2+z^2)\,dx\,dy\,dz=\left|\begin{matrix}\text{uvodimo cilindrične}\\ \text{koordinate}\end{matrix}\right|=\iiint\limits_{W'}(r^2+z^2)\,r\,dr\,d\varphi\,dz=$$

$$=\int_0^{2\pi}d\varphi\int_0^a dr\int_{-\sqrt{3(a^2-r^2)}}^{\sqrt{3(a^2-r^2)}}(r^2+z^2)\,r\,dz=\int_0^{2\pi}d\varphi\int_0^a\left(r^2 z\Big|_{-\sqrt{3(a^2-r^2)}}^{\sqrt{3(a^2-r^2)}}+\frac{z^3}{3}\Big|_{-\sqrt{3(a^2-r^2)}}^{\sqrt{3(a^2-r^2)}}\right)r\,dr$$

$$=\int_0^{2\pi}d\varphi\int_0^a\left(r^2\,2\sqrt{3}\sqrt{a^2-r^2}+\frac{1}{3}\left(3\sqrt{3}\sqrt{(a^2-r^2)^3}+3\sqrt{3}\sqrt{(a^2-r^2)^3}\right)\right)r\,dr=$$

ako ovo izvučemo ispred zagrade

$$=\int_0^{2\pi}d\varphi\int_0^a\left(2\sqrt{3}\,r^2\underline{\underline{\sqrt{a^2-r^2}}}+2\sqrt{3}\,(a^2-r^2)\underline{\underline{\sqrt{a^2-r^2}}}\right)r\,dr=2\sqrt{3}\,a^2\int_0^{2\pi}d\varphi\int_0^a\sqrt{a^2-r^2}\,r\,dr$$

$$=\left|d(a^2-r^2)=-2r\,dr\right|=-\sqrt{3}\,a^2\int_0^{2\pi}d\varphi\int_0^a\sqrt{a^2-r^2}\,d(a^2-r^2)=\ \dots\ =\frac{1}{\sqrt{3}}4\pi a^5$$

rješenje

Izračunati $I = \iiint\limits_{\Omega} \sqrt{x^2+y^2}\, dx\, dy\, dz$ gdje je Ω oblast

$x^2+y^2+z^2 \le 1$ i $x^2+y^2+z^2 \le 2z$.

Rj.

$x^2+y^2+z^2 \le 1$

je unutrašnjost sfere poluprečnika 1 sa centrom u tački $(0,0,0)$

$x^2+y^2+z^2 \le 2z$

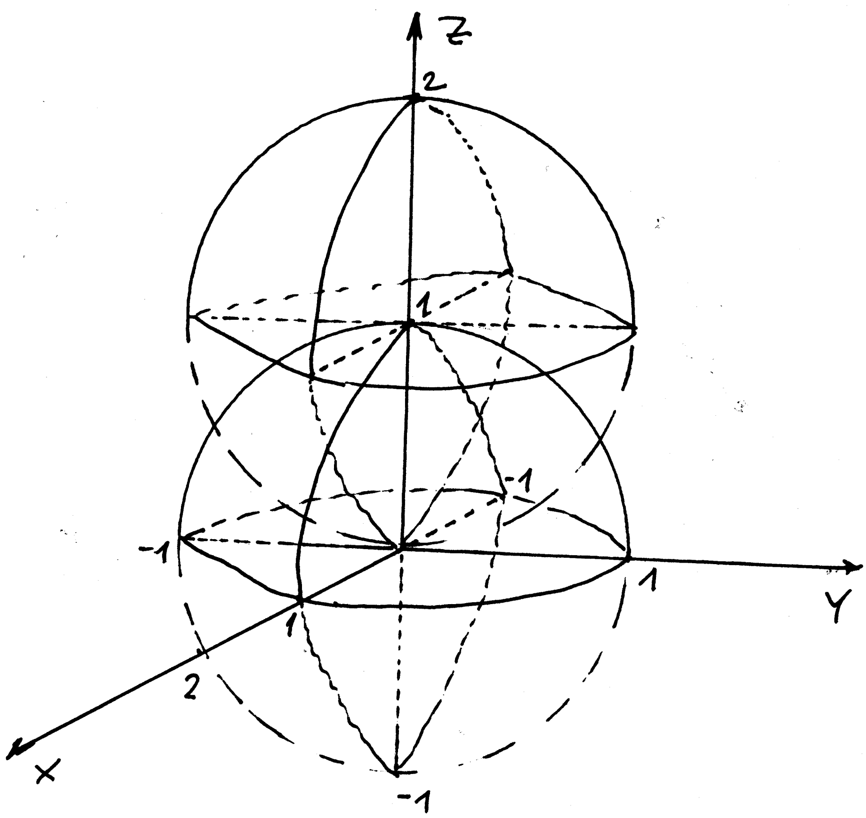
$x^2+y^2+z^2-2z \le 0$

$x^2+y^2+z^2-2\cdot z\cdot 1+1 \le 1$

$x^2+y^2+(z-1)^2 \le 1$

unutrašnjost sfere poluprečnika 1 sa centrom u tački $(0,0,1)$

Skicirajmo dvije date sfere



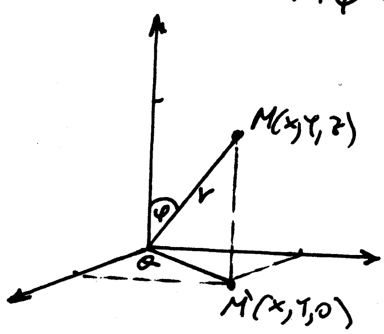
opišimo oblast integracije uz pomoć sfernih koordinata uvodimo smjenu

$x = r\sin\varphi\cos\theta$

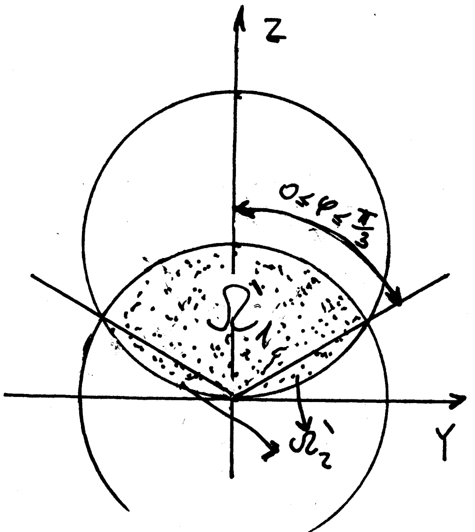
$y = r\sin\varphi\sin\theta$

$z = r\cos\varphi$

$dx\,dy\,dz = r^2\sin\varphi\, dr\, d\varphi\, d\theta$



Napravimo projekciju oblasti na yoz ravan.



$x^2+y^2+z^2 \le 1$

$(r\sin\varphi\cos\theta)^2+(r\sin\varphi\sin\theta)^2+(r\cos\varphi)^2 \le 1$

$r^2 \le 1$

$0 \le r \le 1$

$x^2+y^2+z^2 \le 2z$

$r^2 \le 2r\cos\varphi \quad /:r$

$r \le 2\cos\varphi$

$0 \le r \le 2\cos\varphi$

$0 \le \cos\varphi \le 1 \quad \forall\varphi$

Može biti $2\cos\varphi < 1$ i $2\cos\varphi > 1$.

1° $2\cos\varphi < 1 \Rightarrow \cos\varphi < \frac{1}{2}$ (pa kako je $\cos\varphi > 0$) $\Rightarrow \varphi \in (\frac{\pi}{3}, \frac{\pi}{2})$

$$\Omega_2 : \begin{cases} 0 \le r \le 2\cos\varphi \\ \frac{\pi}{3} \le \varphi \le \frac{\pi}{2} \\ 0 \le \theta \le 2\pi \end{cases}$$

2° $2\cos\varphi > 1 \Rightarrow \cos\varphi > \frac{1}{2}$ (pa kako je $\cos\varphi \le 1$) $\Rightarrow \varphi \in (0, \frac{\pi}{3})$

$$\Omega_1 : \begin{cases} 0 \le r \le 1 \\ 0 \le \varphi \le \frac{\pi}{3} \\ 0 \le \theta \le 2\pi \end{cases}$$

$\Omega = \Omega_1 \cup \Omega_2$ (Ω_1' i Ω_2' su projekcije oblasti Ω na yOz ravan) (vidi sliku)

$$x^2 + y^2 = r^2\sin^2\varphi\cos^2\theta + r^2\sin^2\varphi\sin^2\theta = r^2\sin^2\varphi$$

$$I = \iiint_\Omega \sqrt{x^2+y^2}\,dx\,dy\,dz = \iiint_{\Omega_1} \sqrt{x^2+y^2}\,dx\,dy\,dz + \iiint_{\Omega_2} \sqrt{x^2+y^2}\,dx\,dy\,dz = I_1 + I_2$$

$$I_1 = \iiint_{\Omega_1} \sqrt{r^2\sin^2\varphi}\; r^2\sin\varphi\,dr\,d\varphi\,d\theta = \int_0^{2\pi} d\theta \int_0^1 r^3\,dr \int_0^{\pi/3} \sin^2\varphi\,d\varphi =$$

$$= \int_0^{2\pi} d\theta \int_0^1 r^3\,dr \int_0^{\pi/3} \frac{1}{2}(1-\cos 2\varphi)\,d\varphi = \frac{1}{2}\int_0^{2\pi} d\theta \int_0^1 r^3 \left(\varphi\Big|_0^{\frac{\pi}{3}} - \frac{1}{2}\sin 2\varphi\Big|_0^{\frac{\pi}{3}}\right) = \ldots = \frac{\pi\sqrt{3}}{32} + \frac{\pi^2}{12}$$

$$I_2 = \int_0^{2\pi} d\theta \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \sin^2\varphi \int_0^{2\cos\varphi} r^3\,dr = \ldots = \frac{\pi^2}{12} - \frac{\pi\sqrt{3}}{8}$$

$$I = I_1 + I_2 = \frac{2\pi^2}{12} - \frac{3\pi\sqrt{3}}{32}$$ ← traženo rješenje

Zadaci za vježbu

U zadacima 3547 — 3551 transformisati trojni integral $\iiint\limits_{\Omega} f(x, y, z)\, dx\, dy\, dz$ na cilindrične koordinate ρ, φ, z ($x=\rho\cos\varphi$, $y=\rho\sin\varphi, z=z$), ili na sferne kordinate ρ, θ, φ ($x=\rho\cos\varphi\cdot\sin\theta$, $y=\rho\sin\varphi\sin\theta$, $z=\rho\cos\theta$), a zatim ga svesti na trostruki (sa određenim posebnim granicama integracije).

3547. Ω je oblast u prvom oktantu ograničena cilindrom $x^2+y^2=R^2$ i ravnima $z=0$, $z=1$, $y=x$ i $y=x\sqrt{3}$.

3548. Ω je oblast ograničena cilindrom $x^2+y^2=2x$, ravni $s=0$ i paraboloidom $z=x^2+y^2$.

3549. Ω je deo lopte $x^2+y^2+z^2<R^2$ koji leži u prvom oktantu.

3550. Ω je deo lopte $x^2+y^2+z^2<R^2$ koji leži unutar cilindra $(x^2+y^2)^2=$ $=R^2(x^2-y^2)$ $(x>0)$.

3551. Ω je oblast koja predstavlja zajednički deo dve lopte $x^2+y^2+z^2<R^2$ i $x^2+y^2+(z-R)^2<R^2$.

U zadacima 3552 — 3556 izračunati date integrale prelazeći na cilindrične ili sferne koordinate.

3552. $\int\limits_0^1 dx \int\limits_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dy \int\limits_0^a dz.$

3553. $\int\limits_0^2 dx \int\limits_0^{\sqrt{2x-x^2}} dy \int\limits_0^a z\sqrt{x^2+y^2}\,dz.$

3554. $\int\limits_{-R}^{R} dx \int\limits_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} dy \int\limits_0^{\sqrt{R^2-x^2-y^2}} (x^2+y^2)\, zd.$

3555. $\int\limits_0^1 dx \int\limits_0^{\sqrt{1-x^2}} dy \int\limits_0^{\sqrt{1-x^2-y^2}} \sqrt{x^2+y^2+z^2}\,dz.$

3556. $\iiint\limits_{\Omega} (x^2+y^2)\, dx\, dy\, dz.$ gde je oblast Ω određena nejednakostima $z\geqslant 0$, $r^2\leqslant x^2+y^2+z^2\leqslant R^2$.

3557. $\iiint\limits_{\Omega} \frac{dx\, dy\, dz}{\sqrt{x^2+y^2+(z-2)^2}}$, gde je Ω—lopta $x^2+y^2+z^2<1$.

3558. $\iiint\limits_{\Omega} \frac{dx\, dy\, dz}{\sqrt{x^2+y^2+(z-2)^2}}$, gde je Ω—cilindar $x^2+y^2<1$, $-1<z<1$.

Rješenja

3547. $\int\limits_0^1 dz \int\limits_{\frac{\pi}{4}}^{\frac{\pi}{3}} d\varphi \int\limits_0^R f(\rho\cos\varphi,\ \rho\sin\varphi,\ z)\,\rho\, d\rho.$

3548. $\int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\varphi \int\limits_0^{2\cos\varphi} \rho\, d\rho \int\limits_0^{\rho^2} f(\rho\cos\varphi,\ \rho\sin\varphi.\ z)\, dz.$

3549. $\int\limits_0^{\frac{\pi}{2}} \sin\theta\, d\theta \int\limits_0^{\frac{\pi}{2}} d\varphi \int\limits_0^R f(\rho\cos\varphi\sin\theta,\ \rho\sin\varphi\sin\theta\ \rho\cos\theta)\,\rho^2\, d\rho.$

3550. $\int\limits_{-\frac{\pi}{4}}^{\frac{\pi}{4}} d\varphi \int\limits_0^{R\sqrt{\cos 2\varphi}} \rho\, d\rho \int\limits_{-\sqrt{R^2-\rho^2}}^{\sqrt{R^2-\rho^2}} f(\rho\cos\varphi,\ \rho\sin\varphi,\ z)\, dz.$

Rješenja

3551. $\int\limits_0^{2\pi} d\varphi \int\limits_0^{\frac{R\sqrt{3}}{2}} \rho\, d\rho \int\limits_{R-\sqrt{R^2-\rho^2}}^{\sqrt{R^2-\rho^2}} f(\rho\cos\varphi,\ \rho\sin\varphi,\ z)\, dz$ ili

$$\int\limits_0^{2\pi} d\varphi \int\limits_0^{\frac{\pi}{3}} \sin\theta\, d\theta \int\limits_0^R f(\rho\cos\varphi\sin\theta,\ \rho\sin\varphi\sin\theta,\ \rho\cos\theta)\,\rho^2\, d\rho +$$

$$+\int\limits_0^{2\pi} d\varphi \int\limits_{\frac{\pi}{3}}^{\frac{\pi}{2}} \sin\theta\, d\theta \int\limits_0^{2R\cos\theta} f(\rho\cos\varphi\sin\theta,\ \rho\sin\varphi\sin\theta,\ \rho\cos\theta)\,\rho^2\, d\rho.$$

3552. $\frac{\pi a}{2}$. **3553.** $\frac{8}{9}a^2$. **3554.** $\frac{4}{15}\pi R^5$. **3555.** $\frac{\pi}{8}$.

3556. $\frac{4}{25}\pi(R^5-r^5)$. **3557.** $\frac{2\pi}{3}$.

3558. $\pi\left[3\sqrt{10}+\ln\frac{\sqrt{2}-1}{\sqrt{10}-3}-\sqrt{2}-8\right]$.